SUGGESTED SOLUTIONS TO THE MIDTERM EXAMINATION

SOLUTION TO THE 1ST QUESTION

(a). A PDE is called linear if it has the form $\mathcal{L}u = f$ for given $\mathbb{R}-$linear operator $\mathcal{L}$ and given function f.

(b). A $C^2(\mathbb{R}^2)$ function u satisfying $\Delta u = 0$ is called a harmonic function.

(c). Since

$$\frac{du}{ds}(s, x_0 - s) = \partial_t u(s, x_0 - s) - \partial_x u(s, x_0 - s) = 0,$$

therefore

$$u(t, x) = (t + x)^4.$$

(d). The solution is $u(x, y) = F(x) + G(y)$ where $F, G \in C^1(\mathbb{R})$.

SOLUTION TO THE 2ND QUESTION

(a). Note

$$\begin{aligned}\int_{\mathbb{R}} e^{-x^2} dx =& \sqrt{\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x^2+y^2)} dx dy} \\ =& \sqrt{\int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr d\theta} \\ =& \sqrt{\pi}.\end{aligned}$$

(b). Consider $u_n(t, x) = \frac{1}{n} e^{-n^2 t} \sin nx$, then u_n satisfies

$$\begin{aligned}\partial_t u_n - \partial_x^2 u_n = 0, &\quad t < 0, x \in \mathbb{R}, \\ u_n(0, x) = \frac{1}{n} \sin nx, &\quad x \in \mathbb{R}.\end{aligned}$$

Note

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |u_n(t, x)| = \lim_{n\to} \frac{1}{n} e^{-n^2 t} = \infty,$$

while

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |u_n(0, x)| = 0,$$

therefore the 1D heat equation is not well-posed.

(c). Suppose the intial-boundary value problem has two solutions u_1, u_2 with $u_1 \neq u_2$, denote $v := u_1 - u_2$, then

$$\begin{aligned}\partial_t v - \partial_x^2 v = 0, \quad & (t,x) \in [0,\infty) \times [0,L],\\ v(0,x) = 0, &\\ v(t,0) = 0, v(t,L) = 0. &\end{aligned}$$

Multiplying the above equation by v and integrating the resultant over $[0,\infty) \times [0,L]$,

$$\int_0^L v(t,x)^2 dx + \int_0^\infty \int_0^L |\partial_x v(t,x)|^2 dx dt = 0,$$

which implies that $v \equiv 0$.

(d). Let $v(t,x) = e^{t^7} u(t,x)$, then

$$\begin{aligned}\partial_t v - \partial_x^2 v = 0, \quad & (t,x) \in [0,\infty) \times \mathbb{R},\\ v(0,x) = \phi(x), \quad & x \in \mathbb{R}.\end{aligned}$$

Then

$$v(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y) dy,$$

therefore

$$u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-t^7 - \frac{(x-y)^2}{4t}} \phi(y) dy.$$

SOLUTION TO THE 3RD QUESTION

We claim that for $u \in C^2(B_1) \cap C^1(\overline{B_1})$ with $\Delta u > 0$,

$$\sup_{B_1} u \leq \sup_{\partial B_1} u.$$

Indeed, suppose u attains its maximum in B_1 , then $\Delta u < 0$ which is a contradiction, therefore u must attains its supreme at ∂B_1.

Now for $v \in C^2(B_1) \cap C^1(\overline{B_1})$ with $\Delta v \geq 0$, we consider $v_\varepsilon(x) = v(x) + \varepsilon |x|^2$, then $v_\varepsilon \in C^2(B_1) \cap C^1(\overline{B_1})$ and $\Delta v_\varepsilon > 0$, by the above discussion,

$$\sup_{B_1} v_\varepsilon \leq \sup_{\partial B_1} v_\varepsilon.$$

Since

$$\sup_{B_1} v < \sup_{B_1} v_\varepsilon, \quad \sup_{\partial B_1} v + \varepsilon \geq \sup_{\partial B_1} v_\varepsilon,$$

let ε goes to 0, we have

$$\sup_{B_1} v \leq \sup_{\partial B_1} v.$$

SOLUTION TO THE 4TH QUESTION

(a). By direct computation,

$$\Delta(u^2) = \sum_{i=1}^n \partial_i^2 (u^2) = 2 \sum_{i=1}^n \partial_i (u \partial_i u) = 2 \sum_{i=1}^n |\partial_i u|^2 = 2|\nabla u|^2.$$

(b). By direct computation,

$$\begin{aligned}
&\Delta(\eta^2|\nabla u|^2)\\
=&\sum_{i=1}^{n}\partial_i^2\left(\eta^2\sum_{j=1}^{n}|\partial_j u|^2\right)\\
=&\sum_{i=1}^{n}\partial_i\left(2\eta\partial_i\eta\cdot\sum_{j=1}^{n}|\partial_j u|^2+2\eta^2\sum_{j=1}^{n}\partial_j u\cdot\partial_i\partial_j u\right)\\
=&\sum_{i=1}^{n}\left(2|\partial_i\eta|^2\sum_{j=1}^{n}|\partial_j u|^2+2\eta\partial_i^2\eta\cdot\sum_{j=1}^{n}|\partial_j u|^2+8\eta\partial_i\eta\cdot\sum_{j=1}^{n}\partial_j u\cdot\partial_i\partial_j u+2\eta^2\sum_{j=1}^{n}|\partial_i\partial_j u|^2\right)\\
=&2\eta\Delta\eta|\nabla u|^2+2|\nabla\eta|^2|\nabla u|^2+8\eta\sum_{i,j=1}^{n}\partial_i\eta\cdot\partial_j u\cdot\partial_i\partial_j u+2\eta^2\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2.
\end{aligned}$$

(c). By Hölder's inequality,

$$8\eta\sum_{i,j=1}^{n}\partial_i\eta\cdot\partial_j u\cdot\partial_i\partial_j u\geq -8|\nabla\eta|^2|\nabla u|^2-2\eta^2\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2,$$

then by (b), we have

$$\Delta(\eta^2|\nabla u|^2)\geq(2\eta\Delta\eta-6|\nabla\eta|^2)|\nabla u|^2.$$

(d). Since $\eta\in C_c^2(B_1)$, therefore there exists a constant $C_0>0$ such that

$$|\eta|\leq C_0,\quad |\nabla\eta|\leq C_0,\quad |\Delta\eta|\leq C_0,$$

then by (a) and (d),

$$\Delta(\eta^2|\nabla u|^2+4C_0^2u^2)\geq(2\eta\Delta\eta-6|\nabla\eta|^2+8C_0^2)|\nabla u|^2\geq 0,$$

which implies $\eta^2|\nabla u|^2+Cu^2$ is a subharmonic function where $C=4C_0^2$.

(e). Since $\eta^2|\nabla u|^2+Cu^2$ is a subharmonic function, then

$$\sup_{B_1}(\eta^2|\nabla u|^2+Cu^2)\leq\sup_{\partial B_1}(\eta^2|\nabla u|^2+Cu^2),$$

which implies

$$\sup_{B_{\frac{1}{2}}}|\nabla u|\leq\sqrt{C}\sup_{\partial B_1}u.$$